



Lecture 5

Optical resonators*

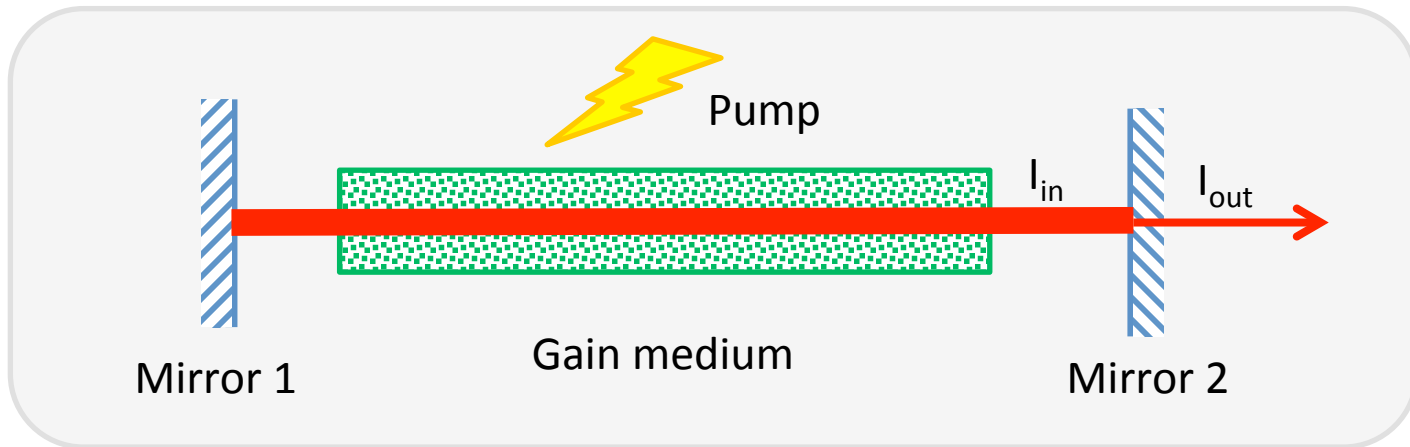
Min Yan

Optics and Photonics, KTH

Reading

- *Principles of Lasers* (5th Ed.): Chapter 5.
- Skip: 5.5.2, 5.5.3, 5.6.2.
- Squeeze*: 5.5.1, 5.6.

Laser



Cavity field: $E(x, y, z, t) = \tilde{E}(x, y, z) \exp\left(j\omega t - \frac{t}{2\tau_c}\right)$

not yet active

Quality: stability, photon lifetime

General properties:

- Partially open
- Long (cm to <1m)

Contents

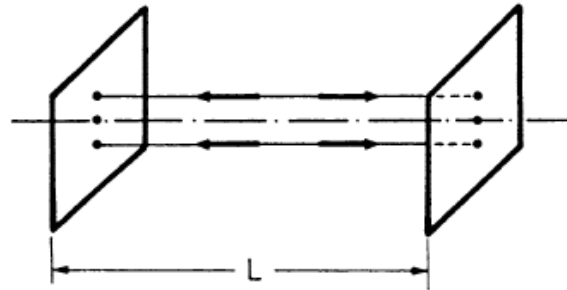
Content	Time
1. Resonator varieties	15'
2. Cavity Q	10'
3. Stability condition	15'
4. Wave-optics mode solutions	35'
5. Unstable resonators	5'
Total:	80'

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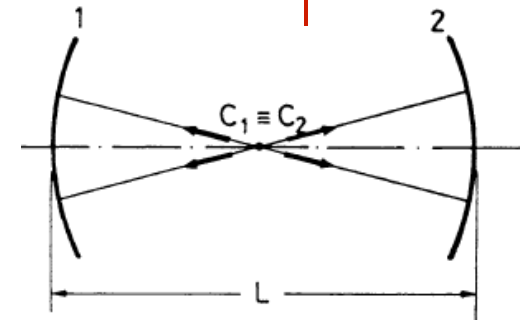
Resonator varieties

- Plane-parallel (FP)

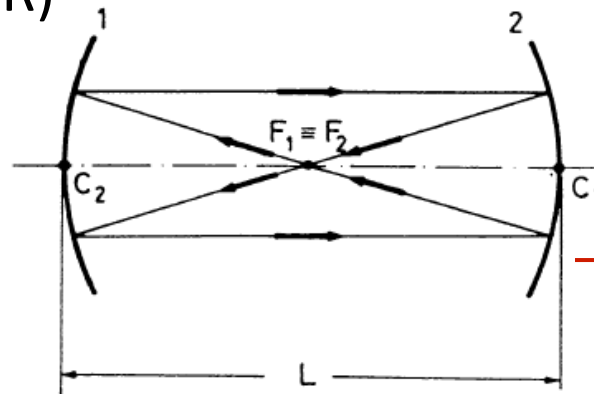


$$\nu_n = n \frac{c}{2L}$$

- Concentric ($L=2R$)



- Confocal ($L=R$)

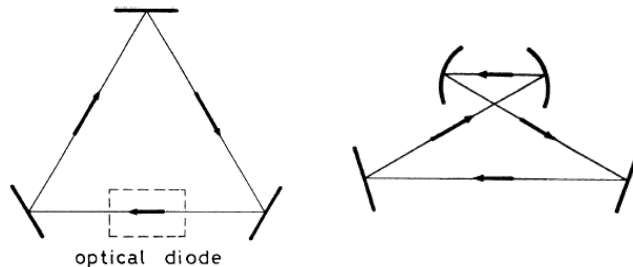


$$\nu_n = n \frac{c}{4L}$$

Others: $R < L < 2R$; different Rs

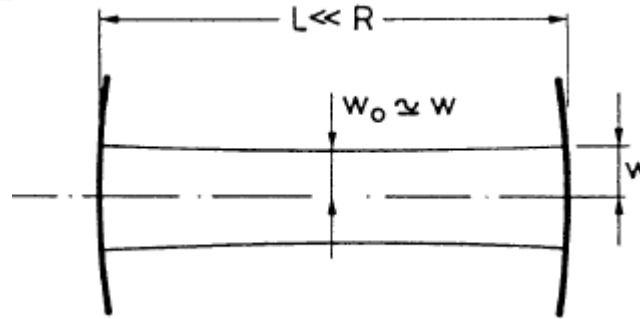
- Stable
- Unstable

- Ring

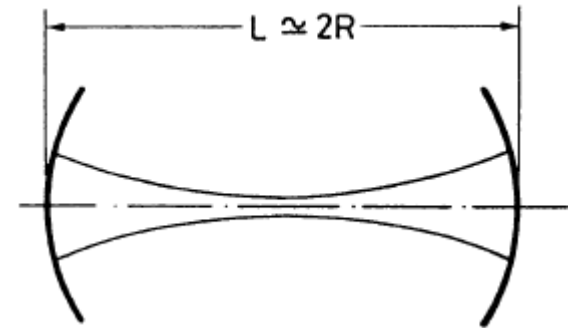


Resonator varieties

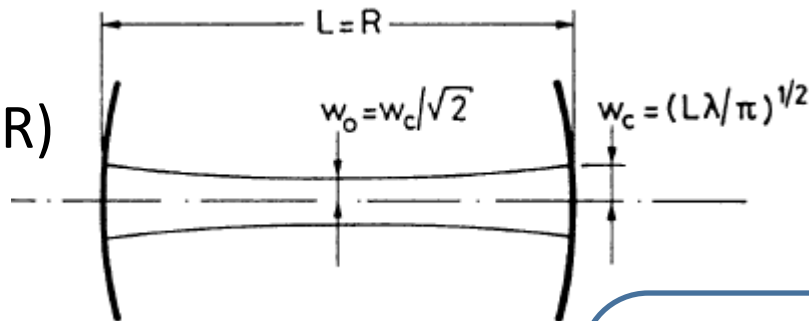
- Plane-parallel (FP)



- Concentric (L=2R)



- Confocal (L=R)



$v_n = ?$
 $q(z) = ?$
 $TEM_{lm} ?$

$$\tilde{E} \propto \exp -jk \left[z + \frac{x^2 + y^2}{2q} \right]$$

$$\frac{1}{q} = \frac{1}{R} - j \frac{\lambda}{\pi w^2}$$

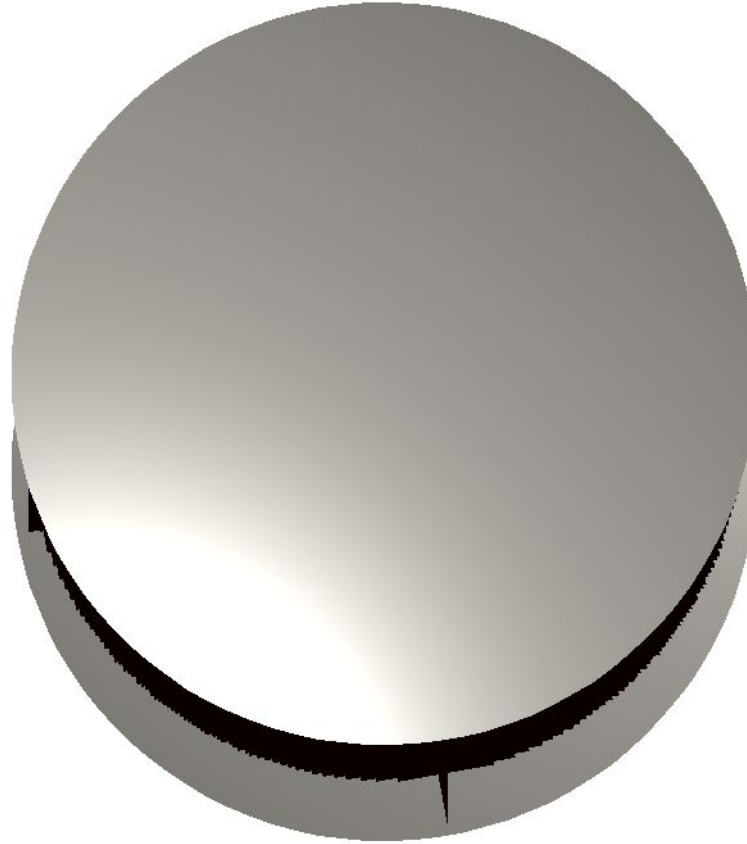
Radius of curvature

Beam spot size

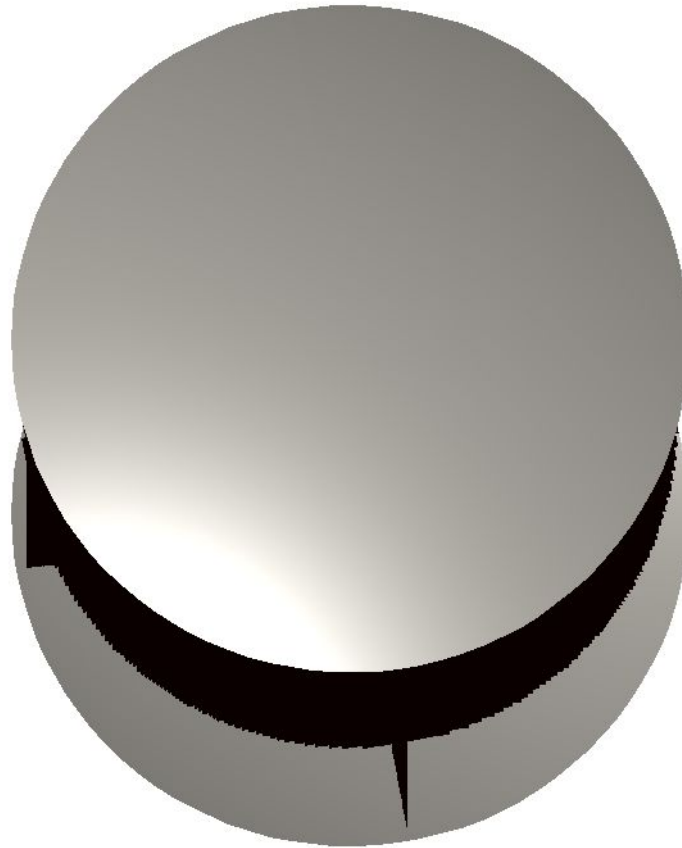
More accurately



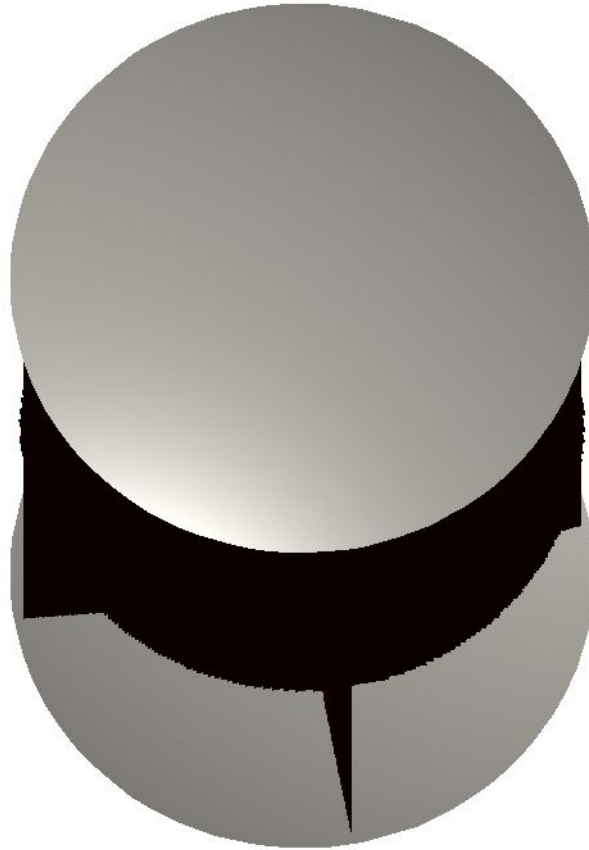
More accurately



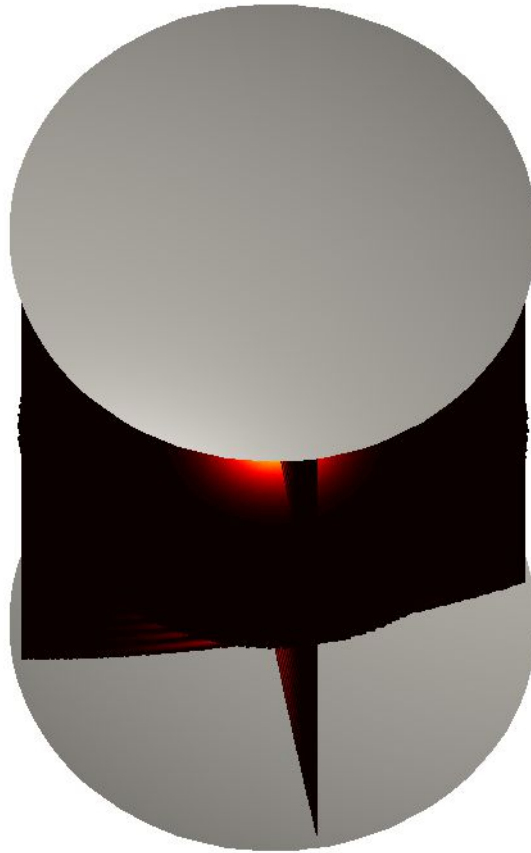
More accurately



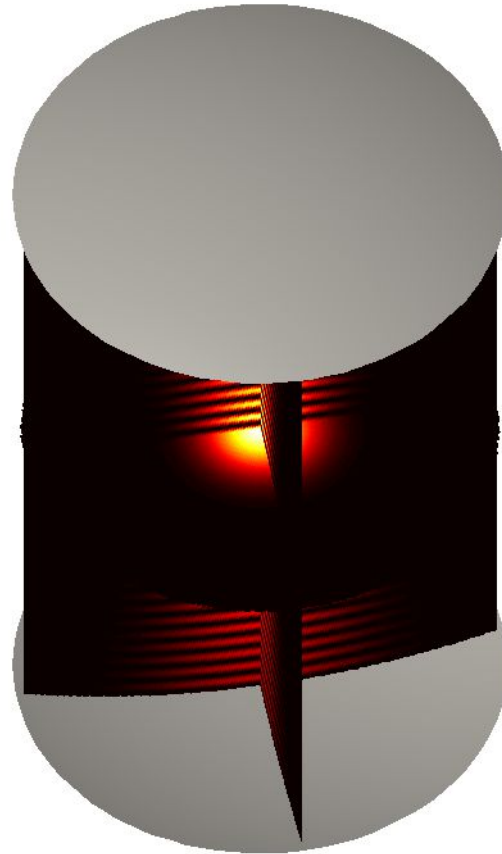
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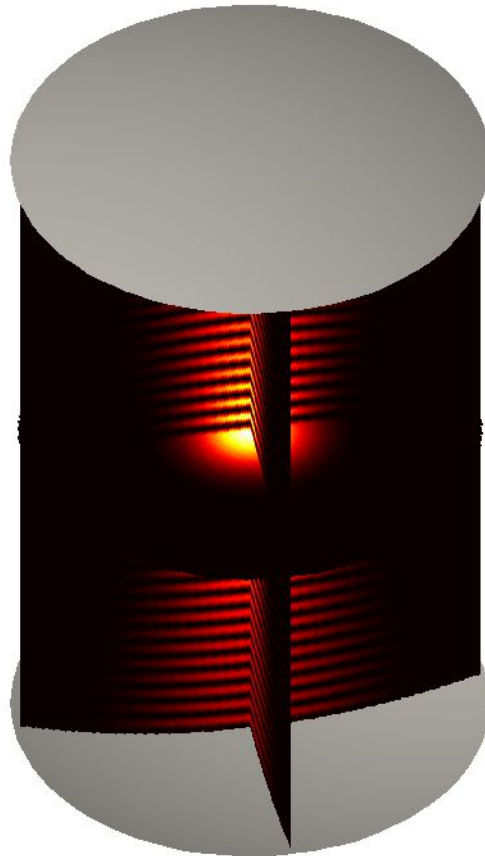
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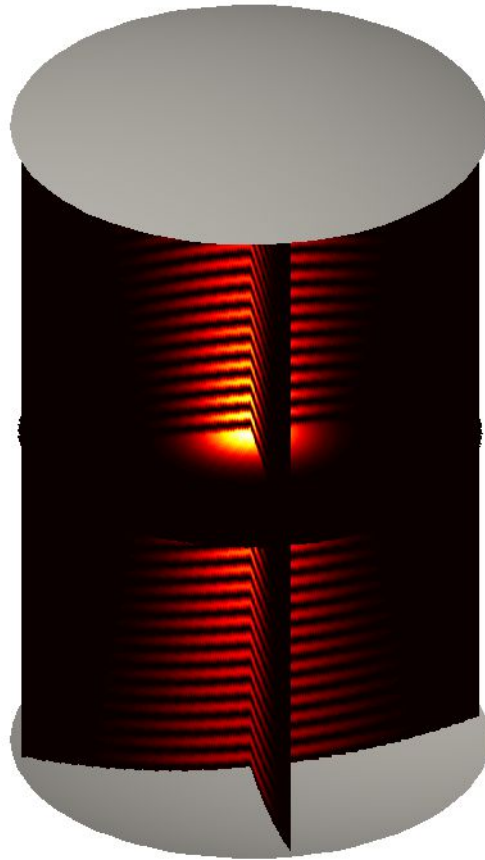
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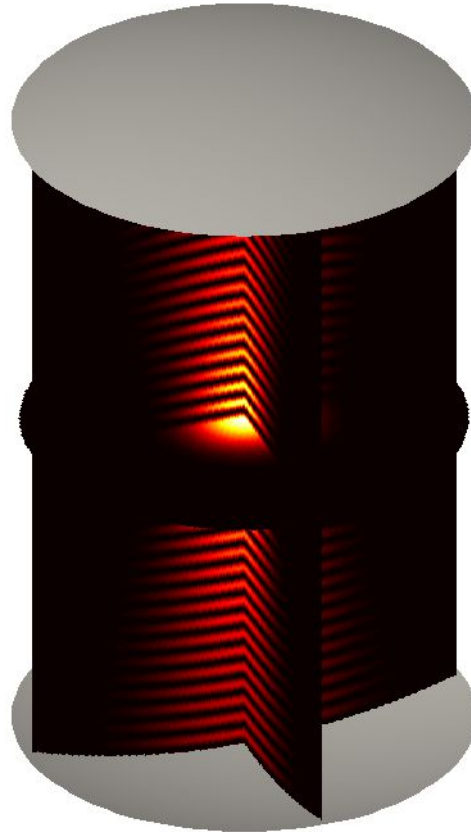
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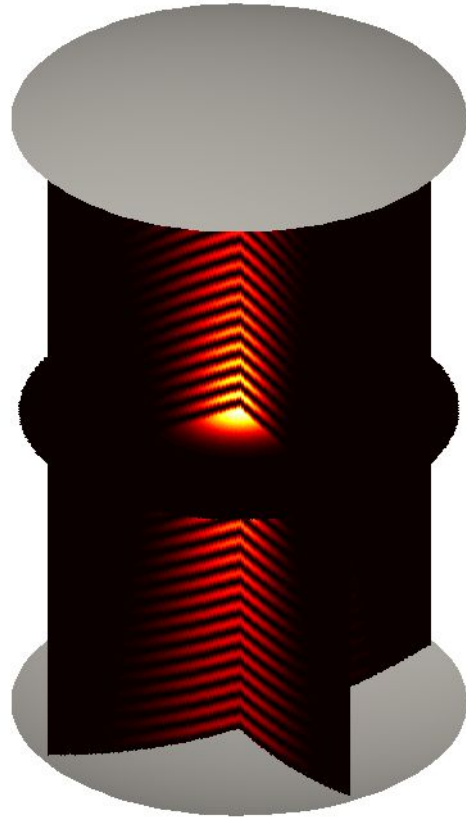
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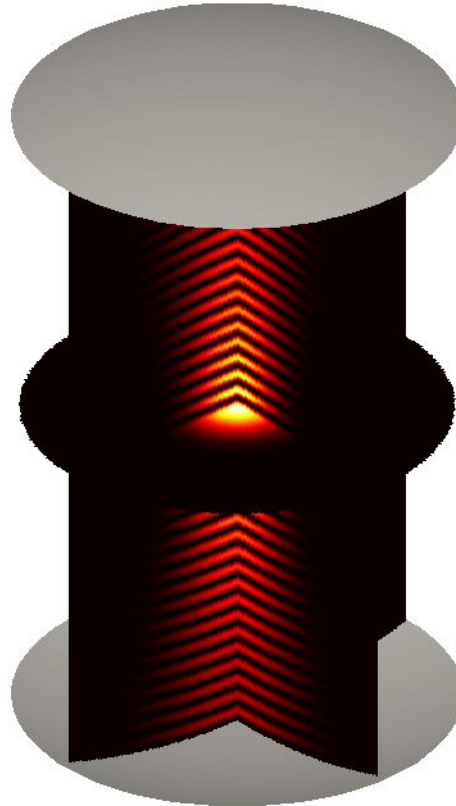
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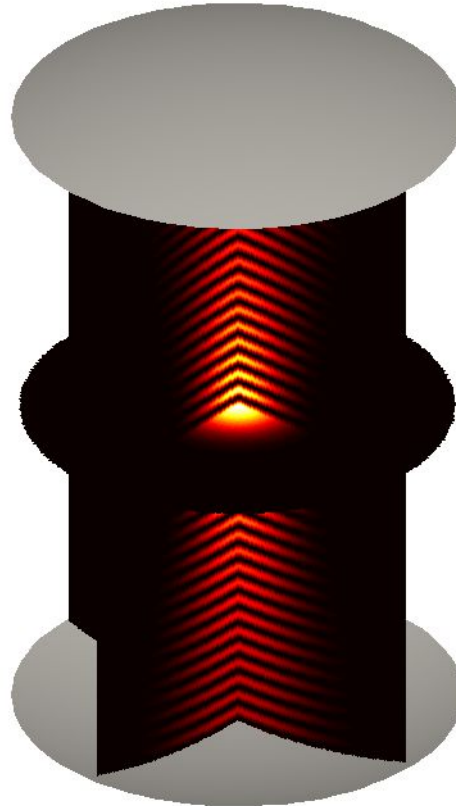
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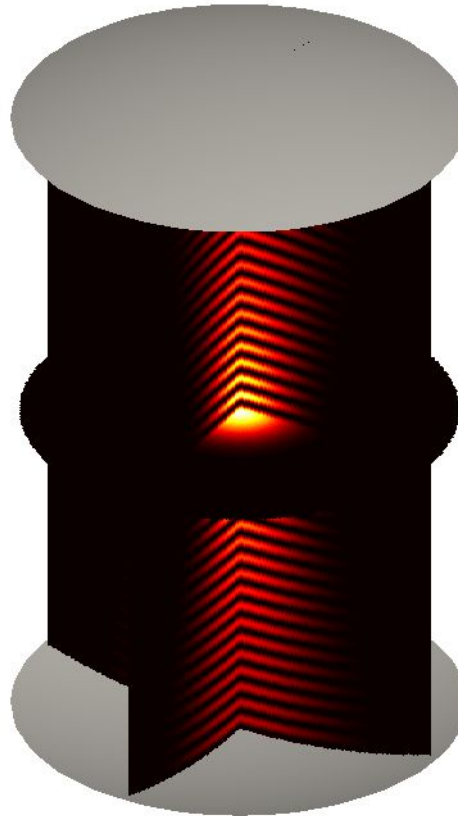
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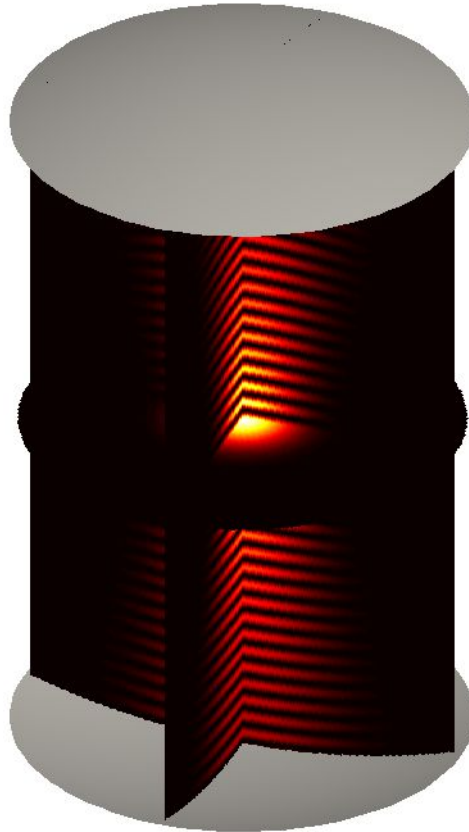
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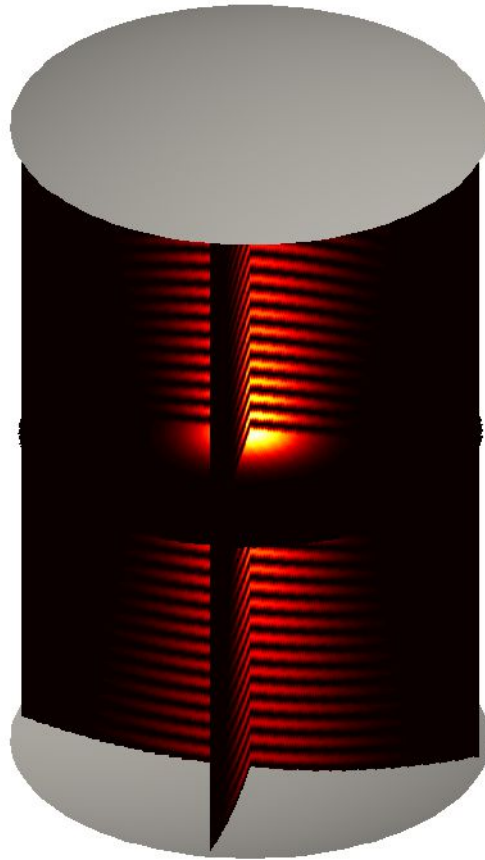
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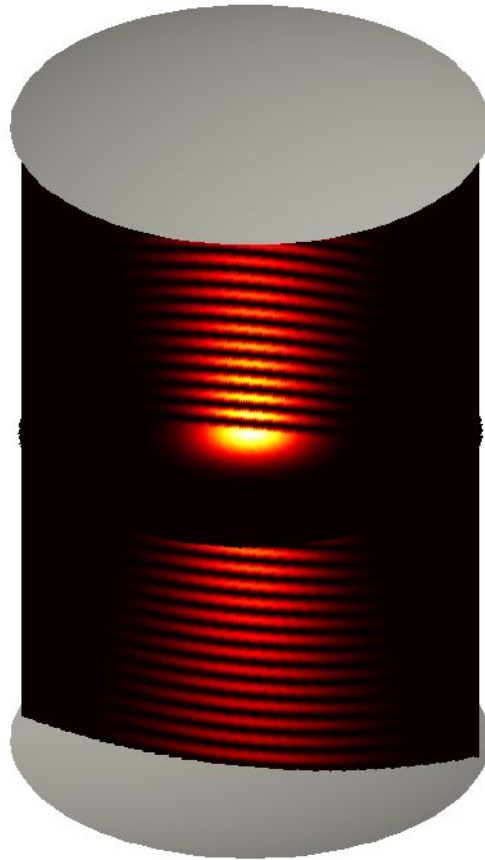
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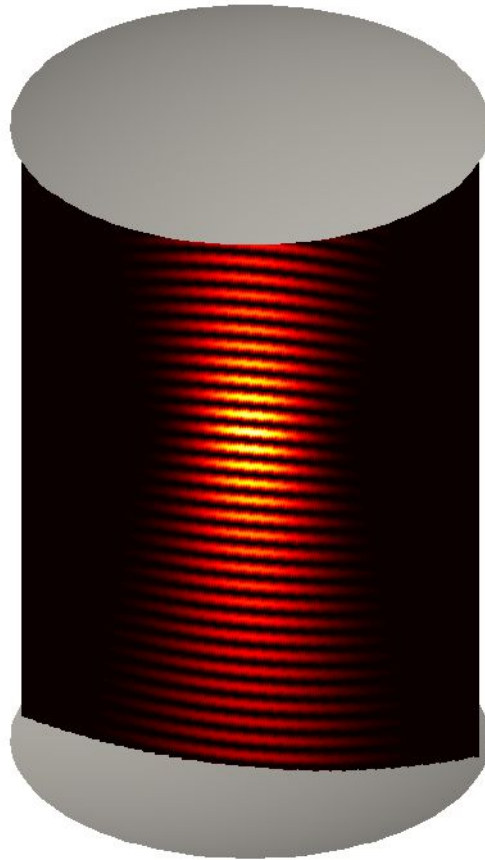
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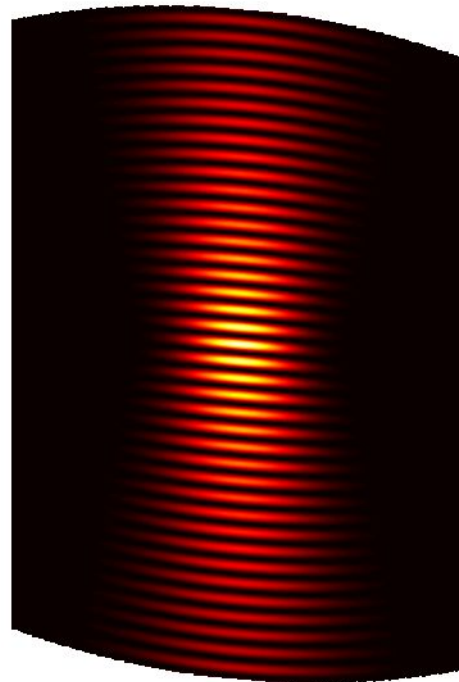
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More accurately



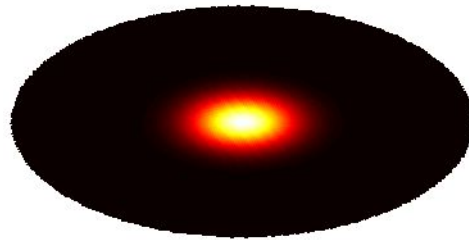
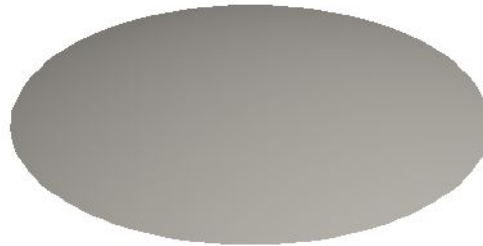
More accurately



- Beam shape
- Phase front
- Resonant freq.
- Decay time

Field: $|E|^2$
 $R=20\mu\text{m}$
 $\lambda_0=1\mu\text{m}$
Confocal

More accurately



- Beam shape
- Phase front
- Resonant freq.
- Decay time

Field: $|E|^2$
R=20 μ m
 $\lambda_0=1\mu$ m
Confocal

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Photon lifetime

Light intensity after m round-trips: $I(t_m) = [R_1 R_2 (1 - T_i)^2]^m I_0$

Photons after m round-trips: $\phi(t_m) = [R_1 R_2 (1 - T_i)^2]^m \phi_0$

Phenomenological assumption: $\phi(t) = \exp\left(-\frac{t}{\tau_c}\right) \phi_0$ $t = m \frac{2L}{c}$

$$\tau_c = -\frac{2L}{c \ln [R_1 R_2 (1 - T_i)^2]} = \frac{L}{c\gamma}$$

γ : single-pass logarithmic cavity loss

Time-dependence of field: $E(t) = \tilde{E} \exp\left(-\frac{t}{2\tau_c} + j\omega t\right)$

By Fourier transform: $\Delta\nu_c = \frac{1}{2\pi\tau_c}$ FWHM of resonance peak (MHz range)

Quality factor

$$Q = 2\pi \frac{\text{energy stored in cavity}}{\text{energy lost in one cycle of oscillation}}$$

$$\begin{aligned} Q &= 2\pi \frac{h\nu\phi}{h\nu \left(-\frac{d\phi}{dt}\right) \frac{1}{\nu}} \\ &= 2\pi\nu\tau_c \\ &= \frac{\nu}{\Delta\nu_c} \end{aligned}$$

Example

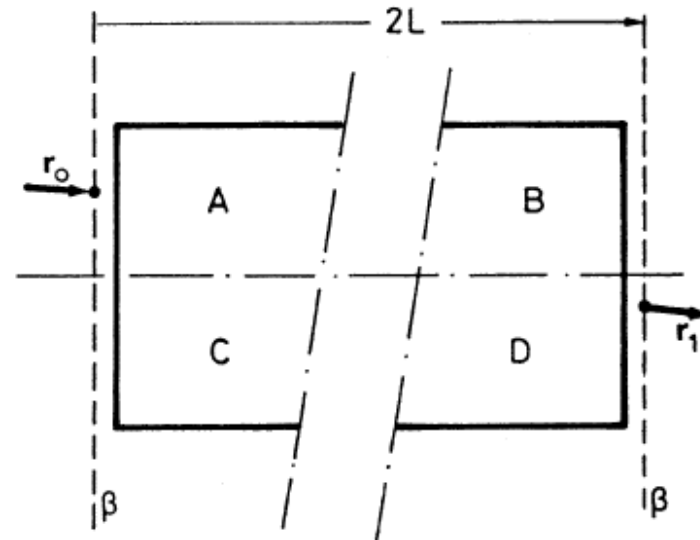
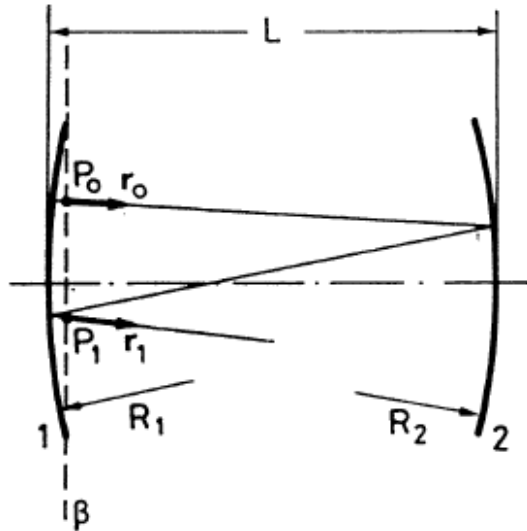
$R_1=R_2=0.98$, $T_i=0$, $L=90\text{cm}$, $\lambda=630\text{nm}$
 $\rightarrow \Delta\nu_c=1.1\text{MHz}$, $Q=4.7\times 10^8$.

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Stability condition

Ray optics



After one roundtrip

$$\begin{vmatrix} r_1 \\ r'_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} r_0 \\ r'_0 \end{vmatrix}$$

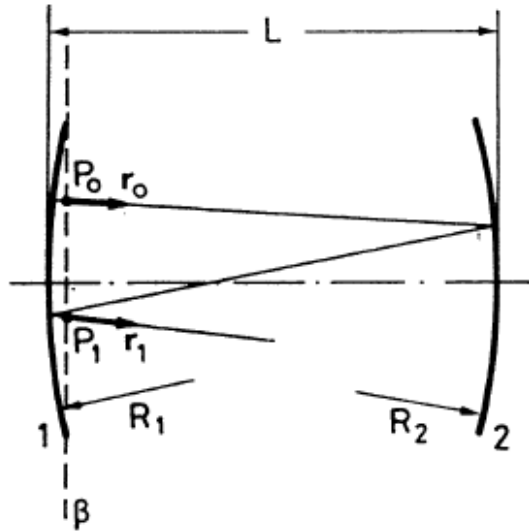
After n roundtrips

$$\begin{vmatrix} r_n \\ r'_n \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}^n \begin{vmatrix} r_0 \\ r'_0 \end{vmatrix}$$

For overall ABCD matrix not to diverge

$$-1 < \left(\frac{A+D}{2} \right) < 1$$

Two-spherical-mirror case Ray optics



ABCD matrix
for spherical mirror:

$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

for free space:

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

After finding round-trip ABCD matrix,

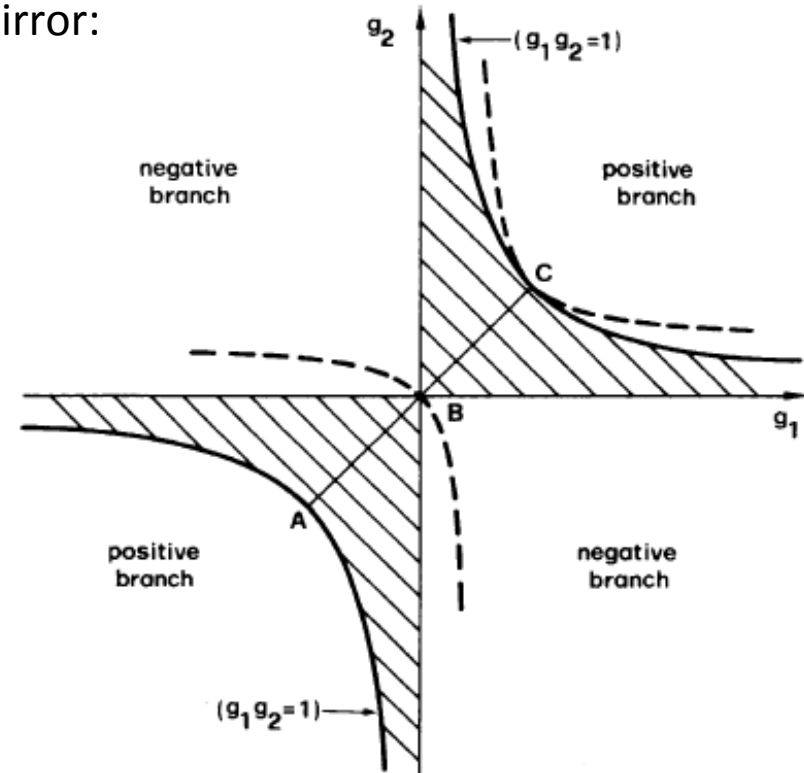
$$\frac{A + D}{2} = 2 \left(1 - \frac{L}{R_1}\right) \left(1 - \frac{L}{R_2}\right) - 1$$

If one defines:

$$g_1 = 1 - (L/R_1)$$

$$g_2 = 1 - (L/R_2)$$

Stability condition: $0 < g_1 g_2 < 1$

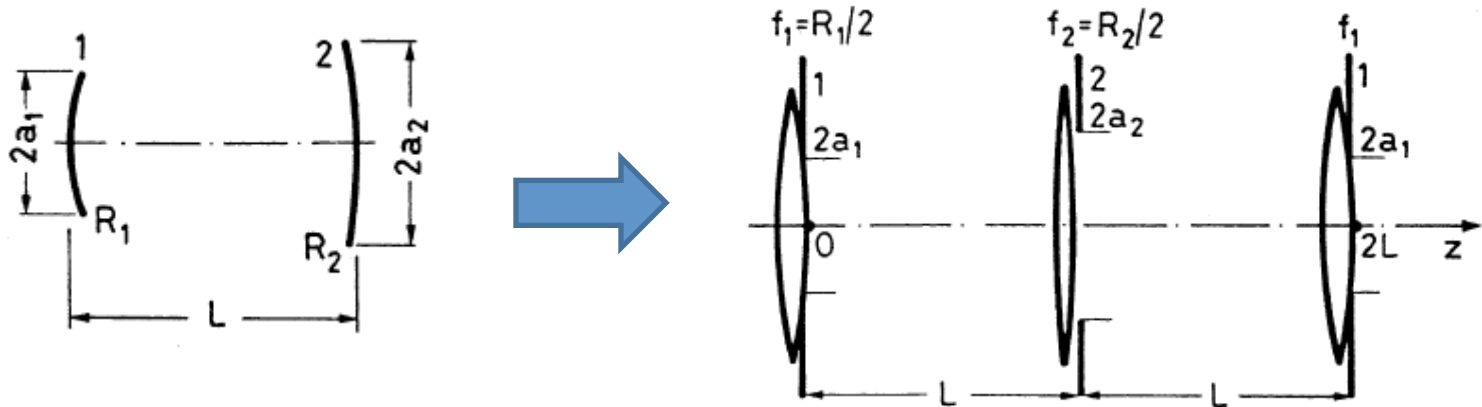


A: concentric
B: confocal
C: planar

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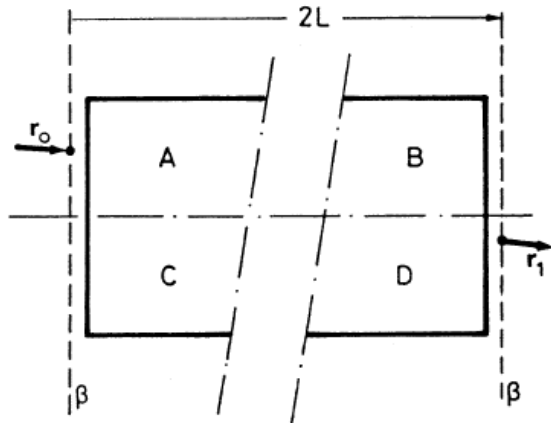
Wave optics



$$\tilde{E}(x, y, 2L) = \exp(-2jkL) \iint_{\infty} \frac{j}{B\lambda} \exp \left[-jk \frac{A(x_1^2 + y_1^2) + D(x^2 + y^2) - 2x_1x - 2y_1y}{2B} \right] \tilde{E}(x_1, y_1, 0) dx_1 dy_1$$

Fresnel-Kirchoff integral

Modes: Infinite aperture



Modes: TEM transverse modes

ABCD law of Gaussian beam propagation

$$q = \frac{A q_1 + B}{C q_1 + D}$$

q must repeat itself after a round trip:

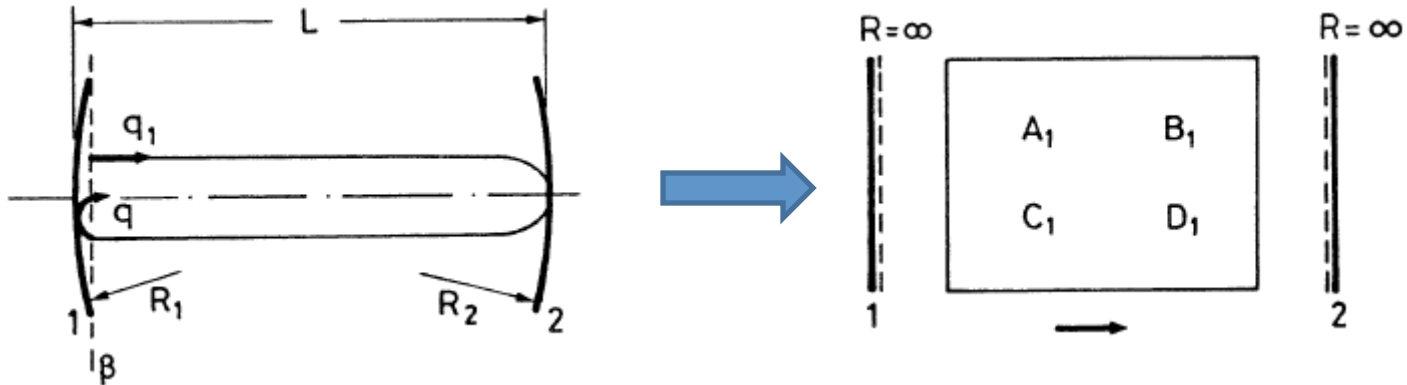
$$q = \frac{Aq + B}{Cq + D} \longrightarrow Cq^2 + (D - A)q - B = 0$$

q must be complex, hence $(D - A)^2 + 4BC < 0$

Since $AD - BC = 1$ $(D + A)^2 < 4$

Stability condition: $-1 < \left(\frac{A + D}{2}\right) < 1$ (re-derived)

Modes: Two-mirror case



Round-trip matrix: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 2A_1D_1 - 1 & 2B_1D_1 \\ 2A_1C_1 & 2A_1D_1 - 1 \end{vmatrix}$

Since $A=D$, from red eq. in previous slide $q = q_1 = j \sqrt{-\frac{B}{C}} = j \sqrt{-\frac{B_1D_1}{A_1C_1}}$

Implication: q_1 is purely imaginary \equiv equiphase surface coincides with mirror 1

Implication 2: beam spot size can be computed based on q_1 .

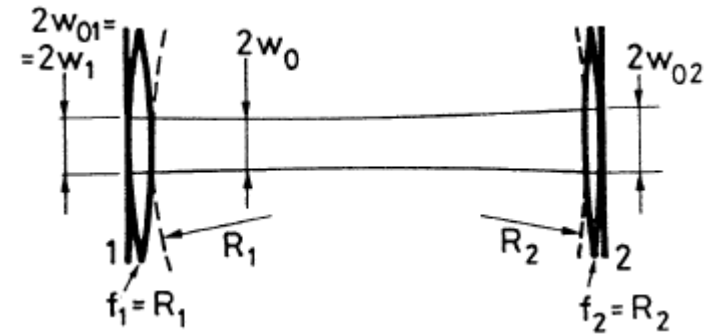
$$\frac{1}{q} = \frac{1}{R} - j \frac{\lambda}{\pi w^2}$$

Beam size

$$\begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix} = \begin{vmatrix} g_1 & L \\ -(1 - g_1 g_2)/L & g_2 \end{vmatrix}$$

$$q_1$$

$$w_1 = \left(\frac{L\lambda}{\pi} \right)^{1/2} \left[\frac{g_2}{g_1 (1 - g_1 g_2)} \right]^{1/4}$$



Special case: symmetric mirrors ($R_1 = R_2 = R$)

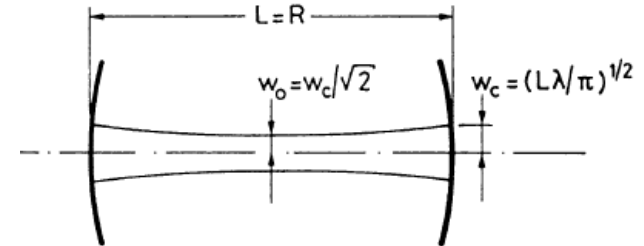
$$w = \left(\frac{L\lambda}{\pi} \right)^{1/2} \left[\frac{1}{1 - g^2} \right]^{1/4}$$

$$w_0 = \left(\frac{L\lambda}{\pi} \right)^{1/2} \left[\frac{1 + g}{4(1 - g)} \right]^{1/4}$$

Beam size: special cases

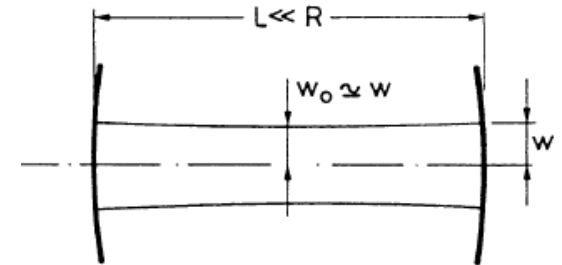
- Confocal ($g=0$):

$$w_c = \sqrt{\frac{L\lambda}{\pi}}, \quad w_0 = \frac{w_c}{\sqrt{2}}$$



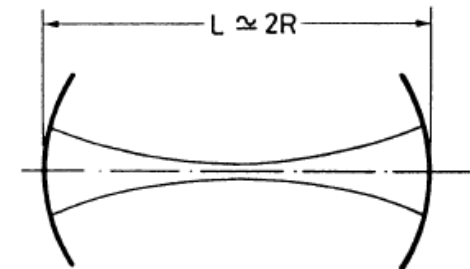
- Near-plane ($g=1-\epsilon$, $\epsilon=L/R$ small):

$$w_c \sqrt[4]{\frac{1}{2\epsilon}}, \quad w_0 \sqrt[4]{\frac{1}{2\epsilon}}$$



- Concentric ($g=-1+\epsilon$, ϵ small):

$$w_c \sqrt[4]{\frac{1}{2\epsilon}}, \quad w_0 \sqrt[4]{\frac{\epsilon}{8}}$$



Example:

$L=1\text{m}$, $\lambda=514\text{nm}$,

- Confocal: $w_c=0.4\text{mm}$; $w_0=0.29\text{mm}$


- Near-plane ($R=10\text{m}$): $w_c=0.61\text{mm}$; $w_0=0.59\text{mm}$

- Concentric ($R=0.5\text{m}$, $\epsilon=0.1$): $w_c=0.61\text{mm}$; $w_0=0.14\text{mm}$

Mode frequencies [keep simple]

Fresnel-Kirchoff integral for paraxial wave through a general ABCD system

$$u(x, y, z) = \frac{j}{B\lambda} \iint_S u(x_1, y_1, z_1) \exp \left[-jk \frac{A(x_1^2 + y_1^2) + D(x^2 + y^2) - 2x_1x - 2y_1y}{2B} \right] dx_1 dy_1$$



That is, if $u(x_1, y_1, z_1) = H_l \left(\frac{\sqrt{2}x_1}{w_1} \right) H_m \left(\frac{\sqrt{2}y_1}{w_1} \right) \exp \left(-jk \frac{x_1^2 + y_1^2}{2q_1} \right)$

then $u(x, y, z) = \left(\frac{1}{A + \frac{B}{q_1}} \right)^{1+l+m} H_l \left(\frac{\sqrt{2}x}{w} \right) H_m \left(\frac{\sqrt{2}y}{w} \right) \exp \left(-jk \frac{x^2 + y^2}{2q} \right)$

After a round trip in cavity $q=q_1$. With round-trip ABCD matrix

$$A + \frac{B}{q} = A - jB \sqrt{-\frac{C}{B}} = \exp(-j\phi)$$

which has a unit amplitude and a phase $\phi = 2 \cos^{-1} \pm \sqrt{A_1 D_1}$

Total phase change for a round trip: $2kL - (1 + l + m)\phi$

which must be $n2\pi$:

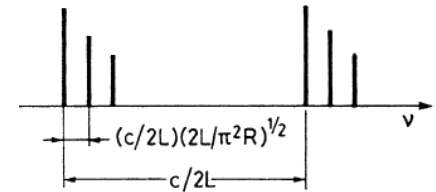
$$\nu_{lmn} = \frac{c}{2L} \left(n + \frac{1 + l + m}{\pi} \cos^{-1} \pm \sqrt{A_1 D_1} \right)$$

Mode frequencies (special cases)

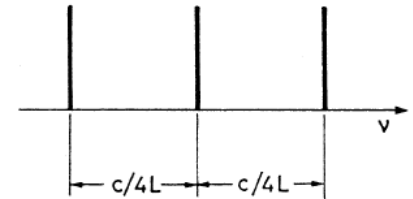
For two-mirror cavities:

$$\nu_{lmn} = \frac{c}{2L} \left(n + \frac{1+l+m}{\pi} \cos^{-1} \pm \sqrt{g_1 g_2} \right)$$

- Near-planar resonator $\nu_{lmn} = \frac{c}{2L} \left[n + \frac{(1+l+m)}{\pi} \left(\frac{2L}{R} \right)^{1/2} \right]$



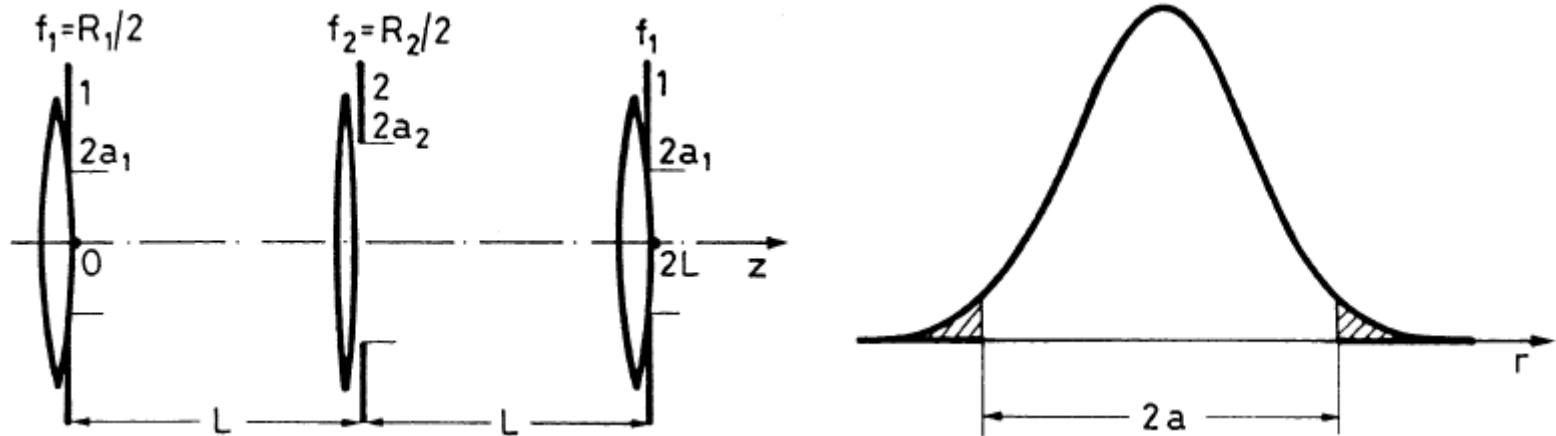
- Confocal resonator $\nu_{lmn} = \frac{c}{4L} [2n + (1+l+m)]$



- Concentric resonator $\nu_{lmn} = \frac{c}{2L} \left(n + \frac{1+l+m}{\pi} \right)$

Finite aperture

Purpose: modal discrimination



Single-transverse-mode condition (approximately):

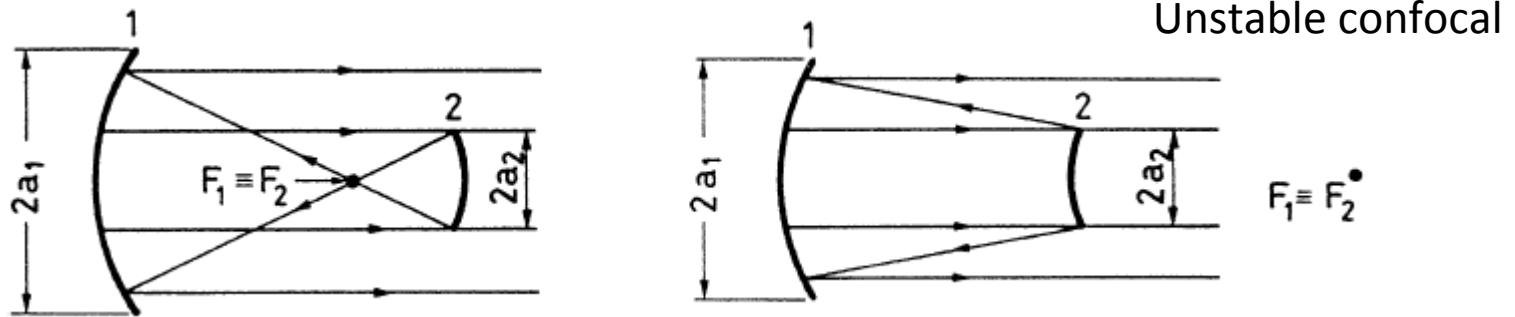
Fresnel number $N = \frac{a^2}{L\lambda} < 2$

For con-focal case: $a < w_c$

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Unstable resonators: Hard-edge



Purpose: large mode area while less modes

Round-trip loss $L_i = \frac{M^2 - 1}{M^2}$

M: field spot magnification
(dependent on g_1 and g_2)

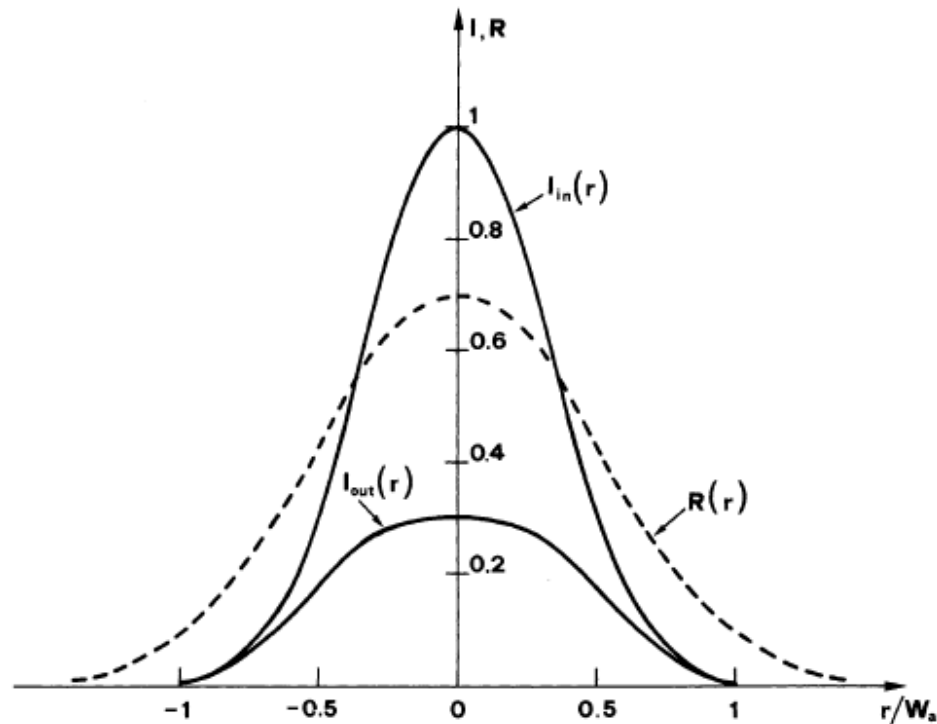
Disadvantage:

- Annular-ring beam shape (doughnut)
- High-order annular rings in beam exist
- Sensitive to perturbation

Unstable resonators: Soft-edge

Solution:

Mirrors with radially-varying reflectivity, with a profile in Gaussian or even super-Gaussian shape.

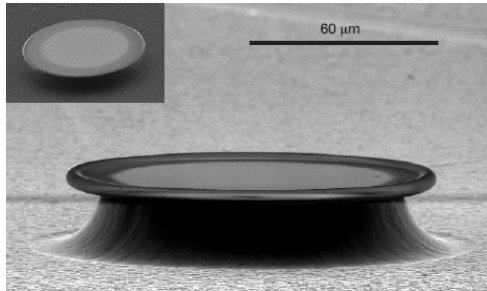


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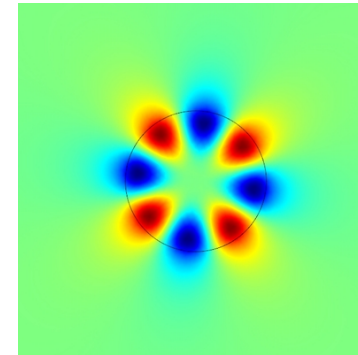
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One more slide

- Fiber cavity
- Microdisk cavity



$Q > 10^8$



- Photonic crystal cavity

